Decay and scattering of solutions to three dimensional nonlinear Schrödinger equations with potentials

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Abstract We prove the decay and scattering of solutions to three dimensional nonlinear Schrödinger with a Schawtz potential. For Rollnik potentials, we obtain time decay and scattering in energy space for small initial data for NLS with pure power nonlinearity $\frac{5}{3} < p < 5$, which is the sharp exponent for scattering. For radial monotone Rollnik potentials, we prove scattering in energy space for $\frac{7}{3} < p < 5$.

1 Introduction

In this paper, we consider the nonlinear Schrödinger equations with potentials

$$\begin{cases} (i\partial_t + \Delta_V)u + \lambda |u|^{p-1}u = 0, \\ u(1) = u_1. \end{cases} \tag{1.1}$$

where $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a complex-valued function, $\Delta_V = \Delta - V$, $\lambda = \pm 1$; $u : I \times \mathbb{R}^d \to \mathbb{C}$. When $\lambda = -1$, we call the equation focusing; when $\lambda = 1$, we call the equation defocusing.

It is a model equation for the single-particle wavefunction in a Bose-Einstein condensate, and sometimes referred to as Gross-Pitaevskii equation. This equation has been intensively studied recently. We first describe some of the works relevant to global wellposedness and scattering in energy space. When $V = 0$, the scattering in energy space for $\frac{4}{d} + 1 < p < \frac{4}{d-2} + 1$ in the defocusing case was proved in J. Ginibre, G. Velo [10] by exploiting Morawetz identities, approximate finite speed of propagation, and strong decay estimates. In energy-critical case $p = \frac{4}{d-2} + 1$, the global wellposedness and scattering was obtained by J. Bourgain [2] for radial data. The radial assumption was removed by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao [4] by establishing frequency-localized interaction Morawetz inequalities. Scattering results for focusing case were obtained via the method of 'compactness-contradiction' pioneered by C. Kenig, F. Merle [16] and followed by R. Killip, M. Visan [17] and B. Dodson [8].

Although some progress on the global wellposedness and scattering has been made when $V \neq 0$, much less is known compared with the $V = 0$ case. Generally speaking, when $V \in L^\infty + L^{\frac{d}{2}+}$, energy-subcritical nonlinear Schr̈dinger equation (1.1) is global well-posed in H^1 in defocusing case and for focusing case the statement holds for small data.(see for example T. Cazenave [6])

When V is harmonic or quadratic, $\lambda = -1$, Oh, Y.G. [20] established the global well-posedness theory in energy space for $1 < p < 1 + \frac{4}{d}$. Then, Carles [5] proved global existence of solutions to defocusing NLS for $1 < p < \frac{4}{d-2} + 1$, and got wave collapse criteria and upper bounds for the blow up time for focusing NLS. R. Killip, M. Visan, X. Zhang [18] proved the global well-posedness for energy critical nonlinearity in defocusing case. When V is inverse-square potential, J. Zhang and J. Zheng [28] establised the well-posedness theory and proved scattering in H^1 for $\frac{4}{d} + 1 < p < \frac{4}{d-2} + 1$.

Another interesting problem is that whether we have scattering or not for $1 < p < \frac{4}{d} + 1$. When $V = 0$, scattering is proved by McKean and Shata [19] for $1 + \frac{2}{d} < p < 1 + \frac{4}{d}$ and small data. Such results were extended to $p \geq 1 + \frac{4}{d}$ or $p < 1 + \frac{4}{d-1}$ when $d \geq 3$ by W. Strauss [23]. When $V \neq 0$, Scipio, Vladimir, Nicola [7] proved the decay estimate for $d = 1$ by the method of invariant norms. The essential ingredient is the decay estimate, namely

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{-\frac{d}{2}}.$$

However, we do not expect scattering even for small data when $1 < p < 1 + \frac{2}{d}$ for (1.1). In fact, W. Strauss [24] proved the zero solution is the only asymptotically free solution when $1 < p \leq 1 + \frac{2}{d}$ for $d \geq 2$ and $1 < p \leq 2$ for $d = 1$ in defocusing case. J. Barab [3] extended it to $1 < p \leq 3$ for $d = 1$. The case when $p = 1 + \frac{2}{d}$ is interesting. In this case, though zero is the only asymptotically free solution as mentioned above, the existence and the form of the modified scattering operator for combined nonlinear term $\lambda|u|^p u + \mu|u|^{p+} u$, where $\mu \neq 0$, was obtained by Ozawa [21] in one dimension and Ginibre and Ozawa [9] for $d \geq 2$.

Our first goal is to prove the scattering in $H^1(\mathbb{R}^3)$ for NLS with radial and decreasing potentials. The proof depends heavily on the interaction Morawetz's inequality. Our second goal is to extend results in McKean and Shatah [19] to the case $V \neq 0$, $d = 3$, namely

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq Ct^{-\frac{3}{2}}. \tag{1.2}$$

And it can also be regarded as an extension of S. Cuccagna, V. Georgiev, N. Visciglia [7] to high dimensions, which to our knowledge is open.

In our arguments, in order to get decay and scattering for small data we assume that

(H1) V is a real-valued Schwartz function satisfying

$$\sup_y \int \frac{|V(x)|}{|x-y|} dx < 2\pi, \tag{1.3}$$

$$\left(\int \frac{|V(x)V(y)|}{|x-y|^2} dx dy \right)^{\frac{1}{2}} < 4\pi. \tag{1.4}$$

In order to prove scattering for large data we assume that

(H2) V is a real-valued Schwartz function satisfying (H1), V is radial, and $V_r \leq 0$ in $r \in (0, \infty)$.

Remark 1.1. *As mentioned above, $V \in L^\infty + L^{\frac{d}{2}+}$ is enough to give a global well-posedness in H^1. Therefore for NLS with V in (H1) or (H2), the global existence of solution is ensured. Moreover we have uniform bound for $\|u(t)\|_{H^1}$.*

Remark 1.2. *(1.4) will lead to that the spectrum $\sigma(\Delta_V) = \sigma_{ac}(\Delta_V) = (-\infty, 0]$, which will be proved in appendix A.*

Remark 1.3. *If u is the solution of (1.1) with initial datum u_0, then $u^\mu = \mu^{-\frac{2}{p}} u(x\mu^{-1}, t\mu^{-2})$, satisfies*

$$i{u^\mu}_t + \Delta u^\mu - \mu^{-2} V(x\mu^{-1}) u^\mu + |u^\mu|^{p-1} u^\mu = 0.$$

So the scaling transformation for V is

$$V(x) \to V^\mu \triangleq \mu^{-2} V(x\mu^{-1}).$$

If the norm in (H1) are not invariant to this transformation, for arbitrary Schwartz potential V, by choosing a proper μ, we can make the norm of V^μ small enough to satisfy the conditions in (H1), then prove scattering for u^μ, and scattering for u follows by taking inverse scaling. However, direct calculation shows all the norm in (H1) are invariant with respect to this transformation, which means that assumptions in (H1) cannot be removed by using the scaling of NLS.

Generally speaking, there are at least two ways to prove scattering. One is to regard the potential term as a perturbation which is easy to work for small regular potentials or time-dependent potentials with decay in time. The other view is to combine the potential term and Δ as the dominant term. In this case we need establish Strichartz estimates for Δ_V and prove the equivalence of the norm $\left\|(-\Delta_V)^{\frac{1}{2}} f\right\|_{L^r}$ and $\left\|(-\Delta)^{\frac{1}{2}} f\right\|_{L^r}$ for some proper r. It applies well to inverse-square and quadratic potentials. In this paper, we take the second way to prove scattering for large data, namely

Theorem 1.1. *Assume $\lambda < 0$, V satisfies (H2), $\frac{7}{3} < p < 5$, then for any $u_0 \in H^1$, there exists $u_+ \in H^1$, such that*

$$\left\|e^{it\Delta_V} u_+ - u(t)\right\|_{H^1} \to 0,$$

as $t \to \infty$.

Following [7], We denote by Σ_s the Hilbert space as the closure of $C_0^\infty(\mathbb{R}^3)$ functions with respect to the norm

$$\|u\|_{\Sigma_s}^2 = \|u\|_{H^s(\mathbb{R}^3)}^2 + \||x|^s u\|_{L^2(\mathbb{R}^3)}^2.$$

In the second part, we extend the results in [7] to dimensions three, namely

Theorem 1.2. *If V satisfies (H1), $\lambda = \pm 1$, $\frac{5}{3} < p < 5$, $s_0 > \frac{3}{2}$, then there exists constants ε_0, C_0, such that for each $0 < \varepsilon < \varepsilon_0$, and $\|u(1)\|_{\Sigma_{s_0}} < \varepsilon$, the solution to (1.1) satisfies the decay estimate (1.2) for $t \geq 1$. Furthermore, there exists $u_+ \in H^1(\mathbb{R}^3)$, such that*

$$\lim_{t\to\infty} \left\| e^{it\Delta_V} u_+ - u(t) \right\|_{H^1(\mathbb{R}^3)} = 0$$

We remark that the decay estimate (1.2) can not hold for general data when $\lambda > 0$. In fact, it is known that there exists solitary wave solution $e^{it}Q(x)$ when $\lambda > 0$, at least for $V = 0$. It is direct to see the L^∞ norm of this soliton solution is invariant respect to t. Moreover, the ranges of s_0, p are sharp in the theorem. In fact, let $Q(x)$ be the solitary solution, direct calculation implies $s > \frac{3}{2}$ is necessary to ensure we cannot arbitrarily reduce the Σ_s norm of Q by using scaling of NLS, which means $s_0 > \frac{3}{2}$ is sharp. $p > \frac{5}{3}$ is also sharp due to the results in [23] as mentioned above.

More importantly, compared with [7] we have no assumptions on the scattering matrix (in one dimension it reduces to transmission coefficient and reflection coefficient), but we need (1.3) and (1.4) to simplify some estimates.

In addition, we need to overcome new difficulties which comes from the high dimensions. In [7] the authors use the scattering theory on the line, and develop an explicit representation formula for $\varphi(-\Delta_V)$ in order to prove the decay estimate (1.2), where φ is a Borel function in R, by scattering matrix(lemma 6.3, [7]). Since we don't have such an analogous in high dimensions, things become difficult, when we try to apply the frame in [7]. However, we find that such a formula is not essential, we can use other method to reach the goal. What we will use as our core are the following well-known formula, where $0 < s < 2$,

$$(-\Delta_V)^{\frac{s}{2}} = c(s)(-\Delta_V) \int_0^\infty \tau^{\frac{s}{2}-1}(\tau - \Delta_V)^{-1} d\tau \tag{1.5}$$

and weighted resolvent estimate,

$$\left\| (\tau - \Delta_V)^{-1} \langle x \rangle^{-N} \right\|_{L^p \to L^p}.$$

Besides, we use a three dimension version distorted Fourier transform defined by expansion of eigenfunctions of Schrödinger operator.

The paper is organized as follows. In section 2, we give the distorted Fourier transform. In section 3, we prove some estimates of the weighted resolvent. In section4, we prove Theorem 1.1. In section 5, we give the Striwartz estimates and prove Theorem 1.2. Some details are collected in Appendix A.

2 The distorted Fourier tramsform

Consider the following Lippmann-Schwinger equation,

$$\varphi(x,k) = e^{i(k,x)} - \frac{1}{4\pi}\int \frac{e^{i|k||x-y|}}{|x-y|}V(y)\varphi(y,k)dy. \tag{2.6}$$

Define

$$\|V\|_R^2 = \int \frac{|V(x)V(y)|}{|x-y|^2}dxdy.$$

Proposition 2.1. *([22], Page 99) Let V be a real-valued function in L^1. Let $H_0 = -\Delta$, on $L^2(\mathbb{R}^3)$ and $H = H_0 + V$ in the sense of quadratic forms. Then, there exists a set $\Im \subset R^+$, which is closed, of Lebesgue measure zero, and such that*

(a) $k^2 \notin \Im$, then there is a unique solution $\varphi(x,k)$ of Lippmann-Schwinger equation (4.1) which obeys $|V|^{1/2}\varphi(x,k) \in L^2$.

(b) $f \in L^2$, then

$$(F_V f)(k) = L.I.M.(2\pi)^{-\frac{3}{2}}\int \overline{\varphi(x,k)}f(x)dx$$

exists.

(c)$f \in D(H)$, then

$$F_V(Hf)(k) = k^2 F_V f(k). \tag{2.7}$$

(d)$R(F_V) = L^2$ and

$$\|F_V f(k)\|_{L^2} = \|P_{ac}(H)f\|_{L^2} \tag{2.8}$$

(e)

$$(P_{ac}(H)f)(x) = L.I.M.(2\pi)^{-\frac{3}{2}}\int (F_V f)(k)\varphi(x,k)dk. \tag{2.9}$$

This proposition was established by Ikebe [13], then extended to $d \geq 4$ by Thoe [25] . P. Alsholm and G. Schmidt [1] completed the whole theory.

Remark 2.1. *When $\|V\|_R < 4\pi$, the set $\Im$ is empty, and the L.I.M. can be understood in principal valued sense. Otherwise, since we assume the operator Δ_V only have absolute-spectrum, we can replace $P_{ac}(H)f$ by f in (2.8) and (2.9).*

Lemma 2.2. *When the Schwatz function V satisfies $\|V\|_R < 4\pi$, then $\|\varphi(x,k)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leq C$.*

Proof. From (2.6), we have

$$\begin{aligned}&|\varphi(x,k)|\\ \leq& 1 + \frac{1}{4\pi}\left\|\frac{V^{1/2}(y)}{|x-y|}V^{1/2}(y)\varphi(y,k)\right\|_{L^1(dy)}\end{aligned}$$

$$
\begin{aligned}
&\leq 1+\frac{1}{4\pi}\left\|\frac{V^{1/2}(y)}{|x-y|}V^{1/2}(y)\varphi(y,k)\right\|_{L^1(|x-y|\geq 1,dy)}+\frac{1}{4\pi}\left\|\frac{1}{|x-y|}V(y)\varphi(y,k)\right\|_{L^1(|x-y|\leq 1,dy)}\\
&\leq 1+\frac{1}{4\pi}\left\|V^{1/2}(y)\right\|_{L^2}\left\|V^{1/2}(y)\varphi(y,k)\right\|_{L^2(dy)}+\frac{1}{4\pi}\left\|\frac{1}{|x-y|}\right\|_{L^2(|x-y|\leq 1,dy)}\|V(y)\varphi(y,k)\|_{L^2(dy)}\\
&\leq 1+C\left\|V^{1/2}(y)\varphi(y,k)\right\|_{L^2(dy)}
\end{aligned}
$$

So it suffices to prove

$$\sup_k\left\|V^{1/2}(y)\varphi(y,k)\right\|_{L^2(dy)}\leq C.$$

Again from (2.6), letting $V^{1/2}(y)\varphi(y,k)=\psi(y,k)$, we have

$$
\begin{aligned}
&\|\psi(x,k)\|_{L^2(dx)}\\
\leq&\|V(x)\|_{L^1(dx)}+\frac{1}{4\pi}\int\left\|\frac{V^{1/2}(x)}{|x-y|}\right\|_{L^2(dx)}|V|^{1/2}(y)\,|\psi(y,k)|\,dy\\
\leq&\|V\|_{L^1}+\frac{1}{4\pi}\left(\int\left\|\frac{V^{1/2}(x)}{|x-y|}\right\|^2_{L^2(dx)}|V(y)|dy\right)^{1/2}\|\psi(y,k)\|_{L^2(dy)}\\
\leq&\|V\|_{L^1}+\frac{1}{4\pi}\|V\|_R\|\psi(y,k)\|_{L^2(dy)}
\end{aligned}
$$

Since $\|V\|_R<4\pi$, we have the desired result.

From the boundness of φ and (2.8), it is easy to see

Corollary 2.3.

$$\|g\|_{L^\infty}\leq C\|F_V g\|_{L^1},$$

and

$$\|F_V\,(f)\|_{L^q}\leq C\|f\|_{L^{q'}},$$

where $2\leq q\leq\infty$.

Lemma 2.4.

$$\|u(t,x)\|_{L^\infty(dx)}\leq t^{-\frac{3}{2}}\,\|u(t,x)\|^{1-\frac{3}{2s}}_{L^2(dx)}\,\||J_V(t)|^{\,s}u(t)\|^{\frac{3}{2s}}_{L^2(dx)},$$

where $|J_V|^s f=M(t)t^s(-\Delta_V)^{\frac{s}{2}}M(-t)f$, $M(t)=\exp(\frac{|x|^2 i}{4t})$.

Proof. The proof is almost the same as Lemma 4.3 in [1]. But for completeness, we give a proof here. It suffices to prove

$$\|u(t,x)\|_{L^\infty(dx)}\leq\|u(t,x)\|^{1-\frac{3}{2s}}_{L^2(dx)}\,\|u(t)\|^{\frac{3}{2s}}_{\dot H^s_V(dx)}.$$

From Lemma 2.4, (2.8) and (2.7), we have

$$\begin{aligned}
&\|u(t,x)\|_{L^\infty(dx)}\\
\leq&\|F_V u(t,k)\|_{L^1(dk)}\\
\leq&\|F_V u(t,k)\|_{L^1(|k|\geq r)}+\|F_V u(t,k)\|_{L^1(|k|\leq r)}\\
\leq&\left\||k|^{-s}\right\|_{L^2(|k|\geq r)}\||k|^s F_V u(t,k)\|_{L^2(|k|\geq r)}+Cr^{\frac{3}{2}}\|F_V u(t,k)\|_{L^2(|k|\leq r)}\\
\leq&C\|u(t,x)\|_{\dot{H}^s}r^{\frac{3}{2}-s}+Cr^{\frac{3}{2}}\|u(t,x)\|_{L^2}.
\end{aligned}$$

Choosing r to make the last two terms equal, we have Lemma 2.4.

3 Estimate of resolvent

First we recall the existence of Green function of the resolvent.

Lemma 3.1. *(Page 102, [22]) There is a measurable function $G(x,y,\tau)$, such that*

$$[(-\Delta_V+\tau)^{-1}\psi](x)=\int G(x,y,\tau)\psi(y)dy. \tag{3.10}$$

$$\begin{aligned}
G(x,y,\tau)&=G_0(x,y,\tau)-\int G_0(x,z,\tau)G(z,y,\tau)V(z)dz\\
G_0(x,y,\tau)&=\frac{e^{-\sqrt{\tau}|x-y|}}{4\pi|x-y|}
\end{aligned} \tag{3.11}$$

Now we establish a basic estimate of the Green function.

Corollary 3.2. *For $G(x,y,\tau)$ in Lemma 3.1 we have,*

$$\sup_{x,y,\tau}|x-y|\,|G(x,y,\tau)|\,e^{\sqrt{\tau}|x-y|}\leq C \tag{3.12}$$

Proof. Letting

$$A(x,y,\tau)=|x-y|\,|G(x,y,\tau)|\,e^{\sqrt{\tau}|x-y|},$$

from (3.11), we have

$$\begin{aligned}
&A(x,y,\tau)\\
\leq&\frac{1}{4\pi}+\int e^{\sqrt{\tau}|x-y|}\,|x-y|\,\frac{e^{-\sqrt{\tau}|x-z|}}{4\pi|x-z|}G(z,y,\tau)V(z)dz\\
\leq&\frac{1}{4\pi}+\int e^{\sqrt{\tau}(|x-z|+|z-y|)}(|x-z|+|y-z|)\frac{e^{-\sqrt{\tau}|x-z|}}{4\pi|x-z|}G(z,y,\tau)V(z)dz
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4\pi} + \int A(z,y,\tau)\frac{1}{4\pi\,|x-z|}V(z)dz + \int A(z,y,\tau)\frac{1}{4\pi\,|y-z|}V(z)dz \\
&\leq \sup_{z,y,\tau}|A(z,y,\tau)|\left(\int V(z)\frac{1}{4\pi\,|y-z|}dz + \int V(z)\frac{1}{4\pi\,|x-z|}dz\right) + \frac{1}{4\pi} \\
&= \sup_{z,y,\tau}|A(z,y,\tau)|\,(I_1+I_2) + \frac{1}{4\pi}
\end{aligned}
$$

From (1.3), we have estimate (3.12).

Lemma 3.3. *For $N \geq 0$ and $1 \leq p \leq \infty$, it holds that*

$$
\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}f\right\|_{L^p} \leq C\frac{1}{|\tau|}\|f\|_{L^p}.
$$

Proof. From estimate (3.12) and Young's inequality, we have

$$
\begin{aligned}
&\left\|(-\Delta_V+\tau)^{-1}\langle x\rangle^{-N}f\right\|_{L^p} \\
\leq &\left\|\int G(x,y,\tau)\langle y\rangle^{-N}f(y)dy\right\|_{L^p(dx)} \\
\leq &C\left\|\int \frac{e^{-\sqrt{\tau}|x-y|}}{|x-y|}\langle y\rangle^{-N}f(y)dy\right\|_{L^p(dx)} \\
\leq &C\left\|\frac{e^{-\sqrt{\tau}|.|}}{|.|}\right\|_{L^1}\|f\|_{L^p} \\
\leq &\frac{C}{|\tau|}\|f\|_{L^p}.
\end{aligned}
$$

This bound is useful when we deal with $\tau > 1$. We need the following Lemma for $0 < \tau < 1$.

Lemma 3.4. *For N large enough, $1 \leq p \leq \infty$, we have*

$$
\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}f\right\|_{L^p} \leq \left(\frac{1}{|\tau|}\right)^{\frac{1}{p}}\|f\|_{L^p}, \tag{3.13}
$$

$$
\left\|\langle x\rangle^{-N}(\tau-\Delta_V)^{-1}f\right\|_{L^p} \leq \left(\frac{1}{|\tau|}\right)^{\frac{1}{p'}}\|f\|_{L^p}. \tag{3.14}
$$

Proof. Notice that (3.14) is the dual version of (3.13), it suffices to prove (3.13). As to prove (3.13), we only need check it for $p = 1$ amd $p = \infty$. $p = 1$ is a simple result of Lemma (3.3), therefore, it remains to prove (3.13) for $p = \infty$. From Corollary 2.3, we have

$$
\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}f\right\|_{L^\infty}
$$

$$
\begin{aligned}
&\leq C\left\|F_V\left((\tau-\Delta_V)^{-1}\langle x\rangle^{-N}f\right)\right\|_{L^1}\\
&=C\left\|\left(\tau+|k|^2\right)^{-1}F_V\left(\langle x\rangle^{-N}f\right)\right\|_{L^1}\\
&\leq C\left\|\left(\tau+|k|^2\right)^{-1}F_V\left(\langle x\rangle^{-N}f\right)\right\|_{L^1(|k|\geq 1)}+C\left\|\left(\tau+|k|^2\right)^{-1}F_V\left(\langle x\rangle^{-N}f\right)\right\|_{L^1(|k|\leq 1)}\\
&\leq C\left\||k|^{-2}\right\|_{L^r(|k|\geq 1)}\left\|F_V\left(\langle x\rangle^{-N}f\right)\right\|_{L^{r'}}+C\left\||k|^{-2}\right\|_{L^p(|k|\leq 1)}\left\|F_V\left(\langle x\rangle^{-N}f\right)\right\|_{L^{p'}}\\
&\leq C\left\|\langle x\rangle^{-N}f\right\|_{L^p}+C\left\|\langle x\rangle^{-N}f\right\|_{L^r}\\
&\leq C\|f\|_{L^\infty}\left\|\langle x\rangle^{-N}\right\|_{L^p}+C\|f\|_{L^\infty}\left\|\langle x\rangle^{-N}\right\|_{L^r}
\end{aligned}
$$

where $2>r>\frac{3}{2}$, $\frac{1}{r'}+\frac{1}{r}=1$, $1<p<\frac{3}{2}$, $\frac{1}{p'}+\frac{1}{p}=1$, as a result $r'>2, p'>2$.

The following Lemma implies the equivalence of $\left\|(-\Delta_V)^{\frac{1}{2}}f\right\|_{L^r}$ and $\left\|(-\Delta)^{\frac{1}{2}}f\right\|_{L^r}$, which will be useful in the proof of scattering for large data.

Lemma 3.5. *For $1\leq r\leq\infty$, we have*

$$
\begin{aligned}
\left\|(-\Delta_V)^{\frac{1}{2}}f\right\|_{L^r}&\leq\left\|(-\Delta)^{\frac{1}{2}}f\right\|_{L^2}+C\|f\|_{L^r}\\
\left\|(-\Delta)^{\frac{1}{2}}f\right\|_{L^r}&\leq\left\|(-\Delta_V)^{\frac{1}{2}}f\right\|_{L^2}+C\|f\|_{L^r}
\end{aligned}
$$

Proof. It easy is to see

$$
(\tau-\Delta_V)^{-1}-(\tau-\Delta)^{-1}=-(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}. \tag{3.15}
$$

Therefore, from (1.5), we have

$$
\begin{aligned}
&(-\Delta_V)^{\frac{1}{2}}f\\
=&c(s)\left(-\Delta_V\right)\int_0^\infty\tau^{-\frac{1}{2}}(\tau-\Delta_V)^{-1}fd\tau\\
=&c(s)\left(-\Delta_V\right)\int_0^\infty\tau^{-\frac{1}{2}}\left((\tau-\Delta_V)^{-1}-(\tau-\Delta)^{-1}\right)fd\tau+c(s)\left(-\Delta_V\right)\int_0^\infty\tau^{-\frac{1}{2}}(\tau-\Delta)^{-1}fd\tau\\
=&c(s)\Delta_V\int_0^\infty\tau^{-\frac{1}{2}}(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}fd\tau+(-\Delta)^{\frac{1}{2}}f+c(s)V\int_0^\infty\tau^{-\frac{1}{2}}(\tau-\Delta)^{-1}fd\tau\\
=&-c(s)\int_0^\infty(\tau-\Delta_V)\tau^{-\frac{1}{2}}(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}fd\tau+c(s)V\int_0^\infty\tau^{-\frac{1}{2}}(\tau-\Delta)^{-1}fd\tau\\
&+c(s)\int_0^\infty\tau^{\frac{1}{2}}(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}fd\tau+(-\Delta)^{\frac{1}{2}}f
\end{aligned}
$$

$$=c(s)\int_0^\infty \tau^{\frac{1}{2}}(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}f\,d\tau+(-\Delta)^{\frac{1}{2}}f$$

From Lemma 3.4 and Lemma 3.3, we have

$$\begin{aligned}
&\left\|(-\Delta_V)^{\frac{1}{2}}f-(-\Delta)^{\frac{1}{2}}f\right\|_{L^r}\\
\le& C\int_0^1 \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}f\right\|_{L^r}d\tau+C\int_1^\infty \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}V(\tau-\Delta)^{-1}f\right\|_{L^r}d\tau\\
\le& C\int_0^1 \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}\langle x\rangle^{2N}V\langle x\rangle^{-N}(\tau-\Delta)^{-1}f\right\|_{L^r}d\tau\\
&+C\int_1^\infty \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}\right\|_{L^r\to L^r}\left\|V(\tau-\Delta)^{-1}f\right\|_{L^r}d\tau\\
\le& C\int_0^1 \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}\right\|_{L^r\to L^r}\left\|\langle x\rangle^{2N}V\langle x\rangle^{-N}(\tau-\Delta)^{-1}f\right\|_{L^r}d\tau\\
&+C\int_1^\infty \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}\right\|_{L^r\to L^r}\|V\|_{L^\infty}\left\|(\tau-\Delta)^{-1}f\right\|_{L^r}d\tau\\
\le& C\int_0^1 \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}\right\|_{L^r\to L^r}\left\|\langle x\rangle^{2N}V\right\|_{L^\infty}\left\|\langle x\rangle^{-N}(\tau-\Delta)^{-1}\right\|_{L^r\to L^r}\|f\|_{L^r}d\tau\\
&+C\int_1^\infty \tau^{\frac{1}{2}}\left\|(\tau-\Delta_V)^{-1}\right\|_{L^r\to L^r}\|V\|_{L^\infty}\left\|(\tau-\Delta)^{-1}\right\|_{L^r\to L^r}\|f\|_{L^r}d\tau\\
\le& C\left\|\langle x\rangle^{2N}V\right\|_{L^\infty}\int_0^1 \tau^{\frac{1}{2}-\frac{1}{r}-\frac{1}{r'}}\|f\|_{L^r}d\tau+C\|V\|_{L^\infty}\int_1^\infty \tau^{\frac{1}{2}-2}\|f\|_{L^r}d\tau\\
\le& C\|f\|_{L^r}.
\end{aligned}$$

Thus we have proved Lemma 3.5.

Since $(-\Delta_V)^{\frac{s}{2}}$ doesn't have a Leibniz rule, we need an estimate of the difference of $(-\Delta_V)^{\frac{s}{2}}$ and $(-\Delta)^{\frac{s}{2}}$.

Lemma 3.6.

$$\left\|(-\Delta_V)^{\frac{s}{2}}f\right\|_{L^2}\le\left\|(-\Delta)^{\frac{s}{2}}f\right\|_{L^2}+C\|f\|_{L^\infty}$$
$$\left\|(-\Delta)^{\frac{s}{2}}f\right\|_{L^2}\le\left\|(-\Delta_V)^{\frac{s}{2}}f\right\|_{L^2}+C\|f\|_{L^\infty}$$

for $\frac{3}{2} < s < 2$.

Proof. Similar to the proof of Lemma 3.5, it suffices to prove

$$\left\| \int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f d\tau \right\|_{L^2} \leq C \|f\|_{L^\infty}.$$

From Lemma 3.4,

$$\begin{aligned}
&\left\| \int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f d\tau \right\|_{L^2} \\
\leq& C \int_1^\infty \tau^{\frac{s}{2}-1} \left\| V(\tau - \Delta)^{-1} f \right\|_{L^2} d\tau + C \int_0^1 \tau^{\frac{s}{2}-\frac{1}{2}} \left\| \langle x \rangle^N V (\tau - \Delta)^{-1} f \right\|_{L^2} d\tau \\
\leq& C \|V\|_{L^2} \int_1^\infty \tau^{\frac{s}{2}-1} \left\| (\tau - \Delta)^{-1} f \right\|_{L^\infty} d\tau + C \left\| \langle x \rangle^N V \right\|_{L^2} \int_0^1 \tau^{\frac{s}{2}-\frac{1}{2}} \left\| (\tau - \Delta)^{-1} f \right\|_{L^\infty} d\tau \\
\leq& C \|V\|_{L^2} \int_1^\infty \tau^{\frac{s}{2}-2} \|f\|_{L^\infty} d\tau + C \left\| \langle x \rangle^N V \right\|_{L^2} \int_0^1 \tau^{\frac{s}{2}-\frac{3}{2}} \|f\|_{L^\infty} d\tau \\
\leq& C \|f\|_{L^\infty},
\end{aligned}$$

here we have used $\frac{3}{2} < s < 2$. Thus we have proved Lemma 3.6.

4 Proof of Theorem 1.1

In M. Goldberg, W. Schlag [11], they proved the following dispersive estimates for Schrödinger Operators.

Proposition 4.1. *Let* $|V(x)| \leq C(1+|x|)^{-\beta}$ *for* $x \in \mathbb{R}^3$, $\beta > 3$. *Assume also that zero is neither an eigenvalue nor a resonance of* $H = -\Delta + V$. *Then*

$$\left\| e^{itH} P_{ac} \right\|_{1 \to \infty} \leq C |t|^{-\frac{3}{2}}.$$

Remark 4.1. *Recall that* ψ *is a resonance if it is a distributional solution of the equation* $H\psi = 0$ *which belongs to the space* $L^2(\langle x \rangle^{-\sigma})$ *for some* σ *but not for* $\sigma = 0$. *Under our assumptions about* V, *we will show no resonance occurs in the appendix A. And as mentioned above,* $P_{ac} = I$, *since the whole spectrum is absolute continuous.*

$\left\| e^{itH} \right\|_{2 \to 2} \leq 1$ *is a quick consequence of the fact* V *is real-valued. And by interpolation we*

have

$$\left\|e^{itH}\right\|_{p'\to p} \leq C|t|^{-\frac{3}{2}(\frac{1}{p'}-\frac{1}{p})},$$

where $\frac{1}{p'}+\frac{1}{p}=1$.

Since we have the dispersive estimates, by the same procedure of proving Strichartz estimates for free Schrödinger operator, we can prove the following Strichartz estimates.

Proposition 4.2. *Let u solves the following equation,*

$$\begin{aligned} i\partial_t u + \Delta u - Vu &= h(x,t) \\ u(0) &= f. \end{aligned}$$

Then

$$\|u\|_{L_t^p L_x^q(R^{3+1})} \leq C\left(\|f\|_{L^2(\mathbb{R}^3)} + \|h\|_{L_t^{a'} L_x^{b'}(R^{3+1})}\right),$$

where (p,q) and (a,b) are Strichartz admissible with $a>2$ and $p>2$, namely

$$\frac{2}{p}+\frac{3}{q}=\frac{3}{2}, \frac{2}{a}+\frac{3}{b}=\frac{3}{2}.$$

Remark 4.2. *Although M. A. Keel, T. Tao [15] have proved the endpoint Strichartz estimates of free Schrödinger operator, as far as we know, the same result for Schrödinger operator is still unknown. But fortunately, we will not use the endpoint estimates in our work.*

Recall Strichartz space defined by

$$\begin{aligned} \|u\|_{S^0(I\times\mathbb{R}^3)} &\triangleq \sup_{(q,r)admissible} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \\ \|u\|_{S^1(I\times\mathbb{R}^3)} &\triangleq \|u\|_{S^0(I\times\mathbb{R}^3)} + \|\nabla u\|_{S^0(I\times\mathbb{R}^3)} \end{aligned}$$

Proposition 4.3. *(Interaction Morawetz estimate) Let u be an H^1 solution to NLS on spacetime slab $I\times\mathbb{R}^3$, then*

$$\int\limits_{I\times\mathbb{R}^3} |u|^4 dxdt \leq C\left\|u\right\|_{H^1}^4.$$

Proof. In the following, we can assume u is Schwartz. Let $a(x,y)=|x-y|$. Suppose $u_1(x,t)$, $u_2(y,t)$ be two solutions to NLS equation with initial data u_0. Then $\phi(x,y,t) \triangleq u_1(x,t)u_2(y,t)$: $\mathbb{R}\times\mathbb{R}^6 \to C$ satisfies

$$i\phi_t + \Delta_{x,y}\phi - (V(x)+V(y))\phi - \left(|u_1|^{p-1}+|u_2|^{p-1}\right)\phi = 0.$$

Define

$$M_a(t) = 2\mathrm{Im} \int\limits_{R^6} \overline{\phi}(x,y) \nabla a(x,y) \cdot \nabla \phi(x,y) dxdy,$$

then

$$\begin{aligned}\partial_t M_a(t) &= 2 \int\limits_{R^6} -\Delta\Delta a |\phi|^2 dxdy + 4 \int\limits_{R^6} a_{jk} \mathrm{Re}\left(\overline{\phi_j} \phi_k\right) dxdy \\ &+ 2 \int\limits_{R^6} \nabla a \cdot \{N, \phi\} \, dxdy - 2 \int\limits_{R^6} |\phi|^2 \nabla\left(V(x) + V(y)\right) \cdot \nabla a dxdy\end{aligned}$$

where $\{f, g\} = \mathrm{Re}\left(f \nabla \overline{g} - g \nabla \overline{f}\right)$, $N = \left(|u_1|^{p-1} + |u_2|^{p-1}\right) \phi$. Since $\{a_{jk}\}$ is semi-positve definite matrix, the second term is nonnegative. We use $j = 1, 2, 3$ for x_j, and $l = 1, 2, 3$ for y_l. Then direct calculation shows

$$\begin{aligned}\left\{\left(|u_1|^{p-1} + |u_2|^{p-1}\right) \phi, \phi\right\}^j &= -\frac{p-1}{p+1} \partial_{x_j} \left(|u_1|^{p-1} |\phi|^2\right) \\ \left\{\left(|u_1|^{p-1} + |u_2|^{p-1}\right) \phi, \phi\right\}^l &= -\frac{p-1}{p+1} \partial_{y_l} \left(|u_2|^{p-1} |\phi|^2\right)\end{aligned}$$

So we have

$$2 \int\limits_{\mathbb{R}^6} \nabla a \cdot \{N, \phi\} \, dxdy = \frac{2(p-1)}{p+1} \int\limits_{\mathbb{R}^6} |\phi|^2 \left(|u_1|^{p-1} + |u_2|^{p-1}\right) \Delta_x a dxdy.$$

Since $\Delta_x a \geq 0$, $-\Delta\Delta a = 16\pi\delta\left(x - y\right)$, $|\nabla a| \leq c$, we have

$$\partial_t M_a(t) \geq 32\pi \int\limits_{\mathbb{R}^3} |u|^4 dx - 2c \int\limits_{\mathbb{R}^6} |u_1(x)|^2 |u_2(y)|^2 \left(|V_r(x)| + |V_r(y)|\right) dxdy.$$

Integrating the above formula, we have

$$2 \sup_t |M_a(t)| + 2c \int\limits_{I \times \mathbb{R}^6} |u_1(x)|^2 |u_2(y)|^2 \left(|V_r(x)| + |V_r(y)|\right) dxdy \geq 32\pi \int\limits_{I \times \mathbb{R}^6} |u|^4 dx.$$

From Hardy's inequality, we have

$$\sup_t |M_a(t)| \leq c \left\|u\right\|_{H^1}^4.$$

It suffices to prove

$$\int\limits_{I \times \mathbb{R}^3} |V_r| |u|^2 dxdt \leq c \|u\|_{H^1} \|u\|_{L^2}.$$

Define operator $A=\partial_r+\frac{1}{r}$ as [28], then

$$\langle [A,\Delta_V]\, u,u\rangle=-\int_{\mathbb{R}^3}V_r|u|^2dx-2\int_{\mathbb{R}^3}\frac{1}{|x|^3}\Delta_{S^2}u\bar{u}dx+c|u\,(t,0)|^2,$$

where $c>0$, and

$$\begin{aligned}
&i^{-1}\partial_t\,\langle Au,u\rangle\\
=&\Big\langle A(\Delta_V u-|u|^{p-1}u),u\Big\rangle-\Big\langle Au,(\Delta_V u-|u|^{p-1}u)\Big\rangle\\
=&\langle [A,\Delta_V u]\, u,u\rangle-\Big(\Big\langle A|u|^{p-1}u,u\Big\rangle-\Big\langle Au,|u|^{p-1}u\Big\rangle\Big)\\
=&-\int\limits_{\mathbb{R}^3}V_r|u|^2dx-2\int\limits_{\mathbb{R}^3}\frac{1}{|x|^3}\Delta_{S^2}u\bar{u}dx+c|u\,(t,0)|^2-\int\limits_{\mathbb{R}^3}|u|^2\partial_r|u|^{p-1}dx\\
=&-\int\limits_{\mathbb{R}^3}V_r|u|^2dx-2\int\limits_{\mathbb{R}^3}\frac{1}{|x|^3}\Delta_{S^2}u\bar{u}dx+c|u\,(t,0)|^2-\frac{p-1}{p+1}\int\limits_{\mathbb{R}^3}\partial_r|u|^{p+1}dx\\
\geq&-\int\limits_{\mathbb{R}^3}V_r|u|^2dx
\end{aligned}$$

Then from Hardy's inequality,

$$\int\limits_{I\times\mathbb{R}^3}|V_r||u|^2dxdt\leq C\sup_t|<Au,u>|\leq C\|u\|_{H^1}\|u\|_{L^2}.$$

Thus we have proved our proposition.

Proposition 4.4. *If V satisfies (H2), then for any $u_0\in H^1$, there exists $u_+\in H^1$, such that*

$$\left\|e^{it\Delta_V}u_+-u(t)\right\|_{H^1}\to 0,$$

as $t\to\infty$.

Proof. Let $\delta>0$ be a small constant to be determined later. And divide $\mathbb{R}$ into subintervals $I_j=[t_j,t_{j+1})$, where $1\leq j\leq L$, $t_{L+1}=\infty$, such that

$$\|u\|_{L^4_{t,x}(I_j\times\mathbb{R}^3)}\leq\delta.$$

From Strichartz estimates and Lemma 3.5, we have

$$\begin{aligned}
&\|u\|_{S^1(I_j\times\mathbb{R}^3)}\\
\leq&C\|u(t_j)\|_{L^2}+C\Big\|(-\Delta_V)^{\frac{1}{2}}u(t_j)\Big\|_{L^2}+C\Big\|(-\Delta_V)^{\frac{1}{2}}|u|^{p-1}u(s)\Big\|_{L_t^{2-\varepsilon}L_x^{\frac{6(2-\varepsilon)}{10-7\varepsilon}}(I_j\times\mathbb{R}^3)}\\
\leq&C\|u(t_j)\|_{H^1}+C\Big\|(-\Delta)^{\frac{1}{2}}|u|^{p-1}u\Big\|_{L_t^{2-\varepsilon}L_x^{\frac{6(2-\varepsilon)}{10-7\varepsilon}}(I_j\times\mathbb{R}^3)}+C\Big\||u|^{p-1}u\Big\|_{L_t^{2-\varepsilon}L_x^{\frac{6(2-\varepsilon)}{10-7\varepsilon}}(I_j\times\mathbb{R}^3)}
\end{aligned}$$

$$
\begin{aligned}
\leq & C\|u(t_j)\|_{H^1}+C\|\nabla u\|_{L_t^{2+\varepsilon}L_x^{\frac{12+6\varepsilon}{2+3\varepsilon}}(I_j\times\mathbb{R}^3)}\|u\|^{p-1}_{L_t^{R_\varepsilon}L_x^{S_\varepsilon}(I_j\times\mathbb{R}^3)}+C\|u\|_{L_t^{2+\varepsilon}L_x^{\frac{12+6\varepsilon}{2+3\varepsilon}}(I_j\times\mathbb{R}^3)}\|u\|^{p-1}_{L_t^{R_\varepsilon}L_x^{S_\varepsilon}(I_j\times\mathbb{R}^3)}\\
\leq & C\|u(t_j)\|_{H^1}+C\|\nabla u\|_{L_t^{2+\varepsilon}L_x^{\frac{12+6\varepsilon}{2+3\varepsilon}}(I_j\times\mathbb{R}^3)}\|u\|^{(p-1)\gamma}_{L^4_{t,x}(I_j\times\mathbb{R}^3)}\|u\|^{(p-1)\alpha}_{L_t^\infty L_x^2(I_j\times\mathbb{R}^3)}\|u\|^{(p-1)\beta}_{L_t^\infty L_x^6(I_j\times\mathbb{R}^3)}\\
&+C\|u\|_{L_t^{2+\varepsilon}L_x^{\frac{12+6\varepsilon}{2+3\varepsilon}}(I_j\times\mathbb{R}^3)}\|u\|^{(p-1)\gamma}_{L^4_{t,x}(I_j\times\mathbb{R}^3)}\|u\|^{(p-1)\alpha}_{L_t^\infty L_x^2(I_j\times\mathbb{R}^3)}\|u\|^{(p-1)\beta}_{L_t^\infty L_x^6(I_j\times\mathbb{R}^3)}\\
\leq & C\|u_0\|_{H^1}+C\delta^{(p-1)\gamma}\|u\|_{S^1(I_j\times\mathbb{R}^3)},
\end{aligned}
$$

where $R_\varepsilon=\frac{(p-1)(4-\varepsilon^2)}{2\varepsilon}$, $S_\varepsilon=\frac{3(p-1)(4-\varepsilon^2)}{2(\varepsilon^2-2\varepsilon+4)}$, $\alpha=\frac{8-6\varepsilon+2\varepsilon^2}{(p-1)(4-\varepsilon^2)}-\frac{1}{2}$, $\beta=\frac{3}{2}-\frac{1}{2}\frac{16+4\varepsilon+4\varepsilon^2}{(p-1)(4-\varepsilon^2)}$, $\gamma=\frac{8\varepsilon}{(p-1)(4-\varepsilon^2)}$. Notice that for a fixed $\frac{7}{3}<p<5$, we can always find a small $\varepsilon>0$ such that $\alpha>0$, $\beta>0$, $\gamma>0$.

A standard continuity argument yields

$$\|u\|_{S^1(I_j\times\mathbb{R}^3)}\leq c\|u_0\|_{H^1(\mathbb{R}^3)}.$$

Sum over all the subintervals, we have

$$\|u\|_{S^1(R\times\mathbb{R}^3)}\leq C\|u_0\|_{H^1(\mathbb{R}^3)}.$$

Define $v(t)=e^{-it\Delta_V}u(t)$, we have

$$v(t)-v(\tau)=-i\int_\tau^t e^{-is\Delta_V}|u|^{p-1}u(s)ds. \tag{4.16}$$

From Strichartz estimates and Lemma 3.5, and standard process of choosing admissible pair exponents, we have

$$
\begin{aligned}
&\left\|\int_\tau^t e^{-is\Delta_V}|u|^{p-1}u(s)ds\right\|_{H^1}\\
\leq & C\left\|\int_\tau^t e^{-is\Delta_V}|u|^{p-1}u(s)ds\right\|_{L^2}+C\left\|\int_\tau^t(-\Delta_V)^{\frac{1}{2}}e^{-is\Delta_V}|u|^{p-1}u(s)ds\right\|_{L^2}\\
\leq & C\left\||u|^{p-1}u(s)\right\|_{L^2}+\left\|(-\Delta_V)^{\frac{1}{2}}|u|^{p-1}u\right\|_{L_t^{2-\varepsilon}L_x^{\frac{6(2-\varepsilon)}{10-7\varepsilon}}([\tau,t]\times\mathbb{R}^3)}\\
\leq & C\,\|u\|^{(p-1)\gamma}_{L^4_{t,x}([t,\tau]\times\mathbb{R}^3)}\|u\|_{S^1([t,\tau]\times\mathbb{R}^3)}
\end{aligned}
$$

Since $\|u\|_{S^1(R\times\mathbb{R}^3)}$ is bounded, the scattering follows.

5 The proof of Theorem 1.2.

In order to prove scattering in H^1, it suffices to prove decay estimate (1.2). In fact from (4.16),

$$\begin{aligned}
&\left\|e^{-it\Delta_V}u(t)-e^{-i\tau\Delta_V}u(\tau)\right\|_{H^1}\\
\le&\left\|\int_\tau^t e^{-is\Delta_V}|u|^{p-1}u(s)ds\right\|_{H^1}\le\int_\tau^t\left\||u|^{p-1}u(s)\right\|_{H^1}ds\le\int_\tau^t\|u(s)\|_{H^1}\|u\|_{L^\infty}^{p-1}ds\\
\le&C\int_\tau^t s^{-\frac{3}{2}(p-1)}ds.
\end{aligned}$$

Therefore, scattering follows.

Thanks to Lemma 2.4, we find it suffices to prove $\sup_t\||J_V|^s u\|_{L^2(dx)}\le C$. In the remaining parts, we devote to prove this.

Recall

$$|J_V(t)|^s\equiv M(t)(-t^2\Delta_V)^{\frac{s}{2}}M(-t) \tag{5.17}$$

where $M(t)$ is the multiplier operator $(M(t)f)(x)=\exp(i|x|^2/4t)f(x)$.

Function $|J_V(t)|^s$ to NLS, we deduce

$$(i\partial_t+\Delta_V)|J_V(t)|^s-it^{s-1}M(t)A(s)M(-t)u+\lambda|J_V(t)|^sF=0, \tag{5.18}$$

where $F=|u|^{p-1}u$. $A(s)$ is the following:

$$A(s)=c(s)\int_0^\infty\tau^{\frac{s}{2}}(\tau-\Delta_V)^{-1}(2V+x\cdot\nabla V)(\tau-\Delta_V)^{-1}d\tau. \tag{5.19}$$

From (5.18) and Strichartz estimates, we have

$$\begin{aligned}
&\sup_t\||J_V|^s u\|_{L^2(dx)}\\
\le&C\||J_V|^s(1)u(1)\|_{L^2}+C\left\|t^{s-1}A(s)M(-t)u\right\|_{L_t^pL_x^r}+C\left\||J_V|^s|u|^{p-1}u\right\|_{L_t^1L_x^2}.
\end{aligned} \tag{5.20}$$

In the following, we will bound the last two terms by $\sup_t\||J_V|^s u\|_{L^2(dx)}$. First, we deal with $A(s)$ term.

Proposition 5.1. *If $\frac{3}{2}<s<\frac{15}{8}$, there exits (p,r) satisfying $1<p<2$, $1<r<2$, (p',r') is an*

admissible pair, and

$$(s-\frac{5}{2})p<-1, \tag{5.21}$$

$$\|A(s)f\|_{L^r}\le C\|f\|_{L^\infty}.$$

Proof. Define $V_1(x)=2V+x\cdot\nabla V$, we have by (5.19) that

$$\|A(s)f\|_{L^r}\le C\int_0^\infty \tau^{\frac{s}{2}}\left\|(\tau-\Delta_V)^{-1}V_1(\tau-\Delta_V)^{-1}f\right\|_{L^r}d\tau.$$

From Lemma 3.4, we have

$$\begin{aligned}
&\left\|(\tau-\Delta_V)^{-1}V_1(\tau-\Delta_V)^{-1}f\right\|_{L^r(dx)}\\
\le&\left\|(\tau-\Delta_V)^{-1}\langle x\rangle^{-N}\right\|_{L^r\to L^r}\left\|\langle x\rangle^{2N}V_1\langle x\rangle^{-N}(\tau-\Delta_V)^{-1}f\right\|_{L^r(dx)}\\
\le&\left(\frac{1}{|\tau|}\right)^{\frac{1}{r}}\left\|\langle x\rangle^{2N}V_1\right\|_{L^r}\left\|\langle x\rangle^{-N}(\tau-\Delta_V)^{-1}f\right\|_{L^\infty}\\
\le&C\left(\frac{1}{|\tau|}\right)^{\frac{1}{r}+1}\|f\|_{L^\infty}.
\end{aligned}$$

Similarly we have

$$\left\|(\tau-\Delta_V)^{-1}V_1(\tau-\Delta_V)^{-1}f\right\|_{L^r(dx)}\le\frac{C}{|\tau|^2}\|f\|_{L^\infty}.$$

Applying the two estimates, we have

$$\|A(s)f\|_{L^r}\le C\left(\int_1^\infty|\tau|^{\frac{s}{2}-2}d\tau+\int_0^1|\tau|^{\frac{s}{2}-1-\frac{1}{r}}d\tau\right)\|f\|_{L^\infty},$$

where we need,

$$\begin{cases}\frac{s}{2}-2<-1\\ \frac{s}{2}-1-\frac{1}{r}>-1\\ 2>s>\frac{3}{2}.\end{cases}$$

Simple calculation shows, if $\frac{3}{2}<s<\frac{15}{8}$, then there exists r and p satisfying all the relations

above and below.

$$\begin{cases} \left(s-\frac{5}{2}\right)p < -1 \\ \frac{2}{p'} + \frac{3}{r'} = \frac{3}{2} \\ \frac{1}{r} + \frac{1}{r'} = \frac{1}{p} + \frac{1}{p'} = 1 \\ p', r' > 2. \end{cases}$$

By Proposition 5.1, Lemma 2.4, we have that

$$\begin{aligned} &\left\| t^{s-1} A(s) M(-t) u \right\|_{L^p(dt) L^r(dx)} \\ \leq & C \left\| t^{s-1} \|u\|_{L^\infty} \right\|_{L^p(dt)} \\ \leq & \left\| t^{s-1} t^{-\frac{3}{2}} \|u(1)\|_{L^2(dx)}^{1-\frac{3}{2s}} \, \| |J_V(t)|^s u(t) \|_{L^2(dx)}^{\frac{3}{2s}} \right\|_{L^p(dt)} \\ \leq & \left\| t^{s-\frac{5}{2}} \right\|_{L^p(dt)} \|u(1)\|_{L^2(dx)}^{1-\frac{3}{2s}} \sup_t \| |J_V(t)|^s u(t) \|_{L^2(dx)}^{\frac{3}{2s}} . \end{aligned} \tag{5.22}$$

While Lemma 3.6, Lemma 2.4 yield

$$\begin{aligned} &\left\| |J_V(t)|^s |u|^{p-1} u \right\|_{L^2(dx)} \leq \left\| (-\Delta_V)^{\frac{s}{2}} M(t) |u|^{p-1} u \right\|_{L^2(dx)} t^s \\ \leq & \left\| (-\Delta)^{\frac{s}{2}} M(t) |u|^{p-1} u \right\|_{L^2(dx)} t^s + C \left\| M(t) |u|^{p-1} u \right\|_{L^\infty(dx)} t^s \\ \leq & \left\| |J|^s |u|^{p-1} u \right\|_{L^2(dx)} + C \left\| |u|^{p-1} u \right\|_{L^\infty(dx)} t^s \\ \leq & C \| |J|^s u \|_{L^2(dx)} \|u\|_{L^\infty}^{p-1} + C \|u\|_{L^\infty}^p t^s \\ \leq & C \| |J_V|^s u \|_{L^2(dx)} \|u\|_{L^\infty}^{p-1} + C \|u\|_{L^\infty}^p t^s \\ \leq & C \, \| |J_V|^s u \|_{L^2(dx)}^{\frac{3(p-1)}{2s}+1} \|u\|_{L^2(dx)}^{(1-\frac{3}{2s})(p-1)} \left(\frac{1}{t^{3/2}} \right)^{p-1} \\ & + C \|u\|_{L^2(dx)}^{(1-\frac{3}{2s})p} \| |J_V|^s u \|_{L^2(dx)}^{\frac{3p}{2s}} \left(\frac{1}{t^{3/2}} \right)^p t^s, \end{aligned}$$

where we have used

$$\left\| |J|^\gamma (|u|^{p-1} u) \right\|_{L^2} \leq \|u\|_{L^\infty}^{p-1} \| |J|^\gamma u \|_{L^2}, 0 < \gamma < \frac{5}{3}, p \geq \frac{5}{3},$$

which comes from Lemma 2.3 in Hayashi N., Naumkin P. [12]. Thus

$$\left\| \left\| |J_V(t)|^s |u|^{p-1} u \right\|_{L^2(dx)} \right\|_{L^1(t \geq 1, dt)}$$

$$
\begin{aligned}
\leq & C\left\|u\right\|_{L^2(dx)}^{(1-\frac{3}{2s})(p-1)} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3(p-1)}{2s}+1} \int_1^{\infty} \left(\frac{1}{t^{3/2}}\right)^{p-1} dt \\
& + C\left\|u\right\|_{L^2(dx)}^{(1-\frac{3}{2s})p} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3p}{2s}} \int_1^{\infty} \left(\frac{1}{t^{3/2}}\right)^{p} t^s dt.
\end{aligned}
$$

If $p > \frac{5}{3}$, for each $s_0 > \frac{3}{2}$, there exists s satisfies $\frac{3}{2} < s < s_0$, such that

$$
-\frac{3}{2}p + s < -1, \frac{3}{2} < s < \frac{15}{8}.
$$

Thus, we have

$$
\begin{aligned}
& \left\| \left\| |J_V(t)|^s |u|^{p-1} u \right\|_{L^2(dx)} \right\|_{L^1(t\geq 1, dt)} \\
\leq & C\left\|u\right\|_{L^2(dx)}^{(1-\frac{3}{2s})(p-1)} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3(p-1)}{2s}+1} + C\left\|u\right\|_{L^2(dx)}^{(1-\frac{3}{2s})p} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3p}{2s}}
\end{aligned}
\tag{5.23}
$$

From (5.22), (5.23), (5.20), we have

$$
\begin{aligned}
& \sup_t \left\||J_V|^s u\right\|_{L^2(dx)} \\
\leq & C\left\||J_V|^s(1)u(1)\right\|_{L^2} + C\left\| |J_V|^s |u|^{p-1} u \right\|_{L^1_t L^2_x} + C\left\|t^{s-1} A(s) M(-t) u\right\|_{L^p_t L^r_x} \\
\leq & C\left\|u(1)\right\|_{\Sigma_s} + C\left\|u(1)\right\|_{L^2(dx)}^{(1-\frac{3}{2s})(p-1)} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3(p-1)}{2s}+1} + C\left\|u(1)\right\|_{L^2(dx)}^{(1-\frac{3}{2s})p} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3p}{2s}} \\
& + C\left\|u(1)\right\|_{L^2}^{1-\frac{3}{2s}} \sup_t \left\||J_V|^s u\right\|_{L^2(dx)}^{\frac{3}{2s}}
\end{aligned}
$$

From Young's inequality and continuous method, we obtain Theorem 1.2.

6 Appendix A

Proposition 6.1. *(D. R. Yafaev [27]) Let real-valued potential V satisfies $|V(x)| \leq C(1+|x|)^{-\rho}$, where $\rho > \frac{5}{2}$. Then $H = -\Delta + V$ has a zero-energy resonance if and only if equation*

$$
-\Delta\psi + V\psi = 0
$$

has a nontrivial solution $\psi(x) \in H^2_{loc} \cap L_2^{-\beta}(\mathbb{R}^3), \beta > \frac{1}{2}$ *such that*

$$\psi(x) = \psi_0 |x|^{-1} + \psi_1(x),$$

where $\psi_1 \in L^2(\mathbb{R}^3)$*. Moreover,*

$$\psi_0 = -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} V(x)\psi(x)dx.$$

Proposition 6.2. *(A. Jensen, T. Kato [14] or D. R. Yafaev [26])*
Let real-valued potential V *satisfies* $|V(x)| \leq C(1+|x|)^{-\rho}$*, where* $\rho > 5$*. Assume that the operator* $H = -\Delta + V$ *has a zero-energy resonance but has no zero eigenvalues. Let* $\psi(x)$ *be the solution in proposition 6.1, where* $\psi_0 = \frac{1}{2\sqrt{\pi}}$*. Then the resolvent of* H *considered as an operator* $R(z) : L_2^{\alpha}(\mathbb{R}^3) \to L_2^{-\alpha}(\mathbb{R}^3)$ *for* $\alpha > \frac{5}{2}$ *admits the following asymptotic expansions as* $z \to 0, z \in \rho(H)$*:*

$$R(z) = iz^{-1/2} \langle \cdot, \psi \rangle \psi + r_0 + iz^{1/2} r_1 + A(z),$$

where $r_j : L_2^{\alpha}(\mathbb{R}^3) \to L_2^{-\alpha}(\mathbb{R}^3), j = 1, 2,$ *are some bounded operators, and*

$$\|A(z)\|_{L_2^{\alpha}(\mathbb{R}^3) \to L_2^{-\alpha}(\mathbb{R}^3)} = o(|z|^{1/2}).$$

Proposition 6.3. *([9], 409 Theorem XIII.21 and its proof)*
Let real-valued potential V *satisfies* $\|V\|_R < 4\pi$*, then* $H = -\Delta + V$ *has only absolute spectrum, particularly no eigenvalues. And for any* $f \in C_0^{\infty}(\mathbb{R}^3)$*, we have*

$$\left| \left\langle (z-H)^{-1} f, f \right\rangle \right| \leq C \tag{6.24}$$

where C *is independent on* $z \in C \backslash [0, \infty)$*, and depends on* f*.*

Proposition 6.4. *Let real-valued Schwartz potential* V *satisfies* $\|V\|_R < 4\pi$*, then* $H = -\Delta + V$ *has no zero-energy resonance.*

Proof. From proposition 6.3, we find H has no zero eigenvalue, if we suppose H has a zero-energy resonance, then from proposition 6.2, $\psi(x)$ is nontrivial, and for a fixed $f \in C_0^{\infty}(\mathbb{R}^3)$, such that $\langle f, \psi \rangle \neq 0$, we have

$$\begin{aligned} &\left\langle (z-H)^{-1} f, f \right\rangle = \langle R(z) f, f \rangle \\ =& iz^{-1/2} |\langle f, \psi \rangle|^2 + \langle r_0 f, f \rangle + iz^{1/2} \langle r_1 f, f \rangle + \langle A(z) f, f \rangle \end{aligned}$$

And we have,

$$|\langle r_i f, f\rangle| \le \|r_i f\|_{L_2^{-\alpha}(\mathbb{R}^3)} \|f\|_{L_2^{\alpha}(\mathbb{R}^3)} \le \|r_i\|_{L_2^{\alpha}(\mathbb{R}^3)\to L_2^{-\alpha}(\mathbb{R}^3)} \|f\|^2_{L_2^{\alpha}(\mathbb{R}^3)}$$
$$|\langle A(z) f, f\rangle| \le \|A(z)\|_{L_2^{\alpha}(\mathbb{R}^3)\to L_2^{-\alpha}(\mathbb{R}^3)} \|f\|^2_{L_2^{\alpha}(\mathbb{R}^3)} \le |z|^{1/2} \|f\|^2_{L_2^{\alpha}(\mathbb{R}^3)}$$

let $z \to 0$, this contradicts with (6.24).

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